The Game of Geography

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1 Introduction

Geography is a two player game where players take turns naming places. Each new place must begin with the last letter of the previous place. The first player who cannot come up with a new place beginning with an appropriate letter loses the game. A chain of places might be: Canada, Antigonish, Halifax, Xanthi Iraq. On a basic level the winner of the game tends to be the player who knows the most place names, but as the players become more matched in their abilities, players seem to hit on two main strategies:

- Try to end your words with the letter "A", forcing your opponent to come up with more words beginning with A,
- Try and use words which end in "X", forcing your opponent to come up with X places.

If we wrote down every place name, we would find that there are mire places that begin with A than end with A, and many more places that X than begin with X. However, there are a limited number of places which end with X, and so we soon hit on a third strategy:

- Avoid ending in letter which begin places ending in X, i.e, saying Antigonish is an unwise decision, since your opponent can immediately follow with Halifax.

We can take this further and develop more strategies, which look several turns ahead, but working with place names is not the most efficient way to view Geography.

While the casual player of Geography is content to deal with Geography as we have described it, the mathematician begins to see some of the fundamental concepts of the game. We are looking at a set of object, namely place names, and the relationships between them. It should now have become apparent that what we are looking at is a graph. Now we must pause and develop some terminology to be able to effectively use graphs to develop strategies for Geography.

1.1 Some Graph Theory

A *Graph* $G = (V, E)$ consists of a set V, of *vertices*, and a relationship between them. We create a diagram, where we draw each vertex as a point, and connect two vertices with an *edge* to represent some sort of mathematical relationship between them. We can define this relationship anyway we want, and we can make these edges *directed* or *undirected*. In an undirected graph, the relationship runs both ways, and in a directed graph the relationship only goes one way. For example, if we define our set $V = \mathbb{Z}$, the natural numbers, we can define an edge runs from u to v if u divides v. Thus there is an edge leading from u to v but not from v to u. This is a directed graph. Or we could take the same V and join u and v by an undirected edge if they are in the same equivalence class modulo 5, this is obviously an undirected graph. We might choose to allow more than one edge in the same direction between two points, or edges which act as loops, beginning and ending at the same point, but for our purposes we will consider Graphs with at most one edge between two vertices in each direction.

In the game of Geography, we define our set of vertices, V to be the set of all place names allowed in the game. We say there is an edge beginning at u and ending at v if the last letter of u is the first letter of v. Now we have a directed graph at the base of our game, and we can use mathematical approaches to find winning strategies in this game. However, before we can talk about using Graph Theory, we must first develop some terminology in order to better understand what it is that we are doing.

First we say that an edge, e is *incident* to a vertex, v, if v is one of its end points. We call the number of vertices, $|V|$, the *order* of G, and the number of edges, $|E|$ the *size*. Two vertices are said to be *adjacent* if they are incident with a common edge. Otherwise, the vertices are said to be *nonadjacent*. We call two edges which are not incident to a common vertex *independent*. For any vertex, v, the *neighborhood* of v, denoted $N(v)$ is the set of all vertices which are adjacent to it. The number of edges which are incident to v is called the degree of v, or $\deg(v)$.

As well, using place names will soon become inefficient, and so we label our vertices, and thus $V = \{0, 1, 2, \cdots, n-1\}$. We call the edge which is incident to vertices u and v (u, v). If we are talking about a directed graph, we say that u is the beginning point and v is the end point, so that $(u, v) \neq (v, u)$. Since there is at most one edge in each direction incident to any two vertices, this ordered pair uniquely defines each edge. A *subgraph*, H of G, is a graph in which all vertices in H are also in G, and all of the edges in H are also in G. Two subgraphs are said to be *independent* if they have no vertices in common. If all of the vertices in G are also in H, we call H a *spanning subgraph*. As well, a *path* is a finite alternating sequence of vertices and edges which connects a series of vertices, and where no vertex is visited more than once. The number of edges in this sequence is the *length* of the path.

There are many variants of Geography, played on different graphs. In some of them, we will allow moves along paths of certain lengths. In this case, we take the *Underlying Graph* and form a new *Augmented Graph* which has the same set of vertices, but u and v are connected by an edge if there is a path of a certain length between them. In this case, we will refer to an edge as having length a if it is in the augmented graph, and the path it represents in the underlying graph is of length a.

We are also concerned with a *matching subgraph* of G, which is a subgraph, M of G in which every vertex has degree 1. Thus each vertex is adjacent to exactly one other vertex, giving a pairing. M is *perfect* if it is a spanning subgraph. Thus each vertex in G is connected to exactly one other vertex. $|V(G)|$ must therefore be even. M is *almost perfect* if all but one of the vertices of G is in M. Then we know that $|V(G)|$ is odd. Any vertex which is not in M is called a *hole*. A *directed matching* is a matching where the endpoint of any edge in the matching is the beginning only of edges which end at the beginning of another matching edge. If an edge of G is not in M, we call it *weak with respect to M*, and if a vertex, v is incident only to weak edges, we call it *weak with respect to M*, or simply *weak*, if it is obvious which M we are talking about. An *M-alternating path* is a path whose edges are alternately weak and not weak. An *M-augmenting path* is an M-alternating path with weak end vertices.

Once we put our game of Geography on a graph, we see that an actual game consists of players taking turns going on a path through the graph, and the player who end up at a vertex which is not adjacent to an unvisited vertex loses the game. The game then is to find a path which gives you the final move. Now that we understand some more about the game, we can start to generalize it. Since there are an almost infinite set of place names, we replace the place names with numbers, and start playing in a finite set. As well, we use a mathematical relation to join edges, such as on a line, where two numbers, u and v are adjacent if $|u - v| = 1$, or on a circle, where our relationship is the same, modulo n, the number of vertices in the circle. However, no matter what our underlying Graph looks like, one thing is apparent. If at the end of the game the path which has been traced out is of an odd length than the first player has won, and if it is of even length than the second player has won. To find the strategies which the players use to ensure the path is of the desired parity, we must understand some basics of Game Theory.

1.2 Some Game Theory

A Game, G is *combinatorial* if there is no chance involved, and there are two players who alternate their moves. Combinatorial games have perfect knowledge, and since there is no chance, there are only a finite amount of possible moves. Thus, any combinatorial game can be analyzed to find out which outcome class the game lies in, thus determining under optimum play who should win the game. A game is called $\mathcal{NP}$-complete if it impossible to figure out who the winner should be without direct calculation of each possibility. If a game is played under *normal* rules, the first person who cannot move is the winner. If there is at least one move in the game, this means that the last person to move is the winner. Most games are played under normal rules, but games can also be played with *misère* rules, which means that the first player who cannot move is the winner.

A game, G, has a *Grundy Value*, $\mathcal{G}$, which lies in one of four outcome classes, $\mathcal{R}$, $\mathcal{L}$, $\mathcal{N}$, or $\mathcal{P}$. These stand for Right, Left, Previous, and Next, respectively. If G lies in $\mathcal{R}$, it means that $\mathcal{G}(G) = \mathcal{R}$ and there is a winning strategy for Right which Left cannot counter, whether Right goes first or second. For the game of Geography, this category, as well as $\mathcal{L}$, which is defined analogously for Left, is irrelevant. Geography is an *impartial Game*, which means that at any position the moves are exactly the same for both Right and Left players. A game such as Chess, however is called *Partizan* which is the opposite of Impartial, because there are positions in which White and Black have very different options. Thus, for our purposes we will discard the concept of Right and Left. Instead we will concentrate on the first and second players, whom we shall call Allan and Beatrice, respectively.

Allan has a *strategy* if knows what move he should play to counter any move of Beatrice's. If Allan has a winning strategy which will allow him to win the game no matter how Beatrice plays, than we say that the game G is in N, or $\mathcal{G}(G) = \mathcal{N}$. If Allan does not have a winning strategy, Beatrice must have one, and so we say that the Game is in $\mathcal{P}$, so $\mathcal{G}(G) = \mathcal{P}$. At each individual term, however, we say that $G \in \mathcal{P}$ if the player who has gone Previous has the winning strategy, and $G \in \mathcal{N}$ if the player whose turn is Next has the winning strategy. By developing mathematics for these classes, we can play multiple games at once, and be able to find the winner. As well, there are Grundy Values in each of the outcome classes, but for now we will not be needing these.

Now we have plenty of tools at our disposal to begin analyzing specific games of Geography, drawing on both Game Theory and Graph Theory, and we are ready for the formal definition of the Game, which draws on both Graphs and Games.

2 Formal Definition

The game of Geography is an impartial combinatorial game played on an underlying graph, which is either directed or undirected. A marker is initially placed on one of the vertices, and players take turns moving the marker along paths of a certain length to another vertex. A vertex can be visited only once, thus making the game finite, and normal rules apply, so the first player who cannot move loses the game. A vertex is considered to be visited only if a marker is there at the end of a turn. If the graph is directed, then the players must move in the specified direction, but on an undirected graph they may move in either direction. As well, a move set consisting of Natural numbers $A = \{a_1 \ldots, a_n\}$ is chosen, and then each move must travel on paths of length a_i, where $a_i \in A$. Thus a player can move from vertex j to vertex $j + a_i$ (sometimes modulo n, depending on the shape of the underlying graph). In this case, we build the augmented graph of the game, G, which consists of all of the vertices from our original graph, and the new edges represent any legal move which can be made. From now on, we will say that a game G has n vertices, (or m if we are referring to multiple games), and we will refer to both a Game and its augmented Graph as G.

Now, we set some terminology which we use throughout this paper. We will deal with move sets of at most two, so we will refer to the game $G = n; \{a, b\}$ as the game with n vertices and move set $\{a, b\}$. When it is obvious from the context what the move set is we will refer to a specific game as n. There are many variants of Geography, depending on the specified move set, the underlying graph, and whether the graph is directed or undirected. For all of these games the most obvious strategy is a *matching strategy*, in which there is a subgraph of G with a perfect or almost perfect matching. As we shall see, the outcome of a game which has a perfect matching is in N. However, a matching strategy cannot always be applied. One of the purposes of this paper is to explore in specific games exactly when a perfect matching can be applied, and when it cannot. As well, finding almost-perfect matchings in odd games gives rise to skolem sequences, which are also important.

3 Matching Strategies

To understand how to use matchings to win a game, we must first understand what a matching is, and when it is perfect or almost perfect. For this we need one more definition, that of a *maximum matching*. A matching, M is maximum if there is no other matching subgraph of G which contains more connected vertices than M. Obviously, if there is a perfect matching, than this is also maximum matching, since all vertices are in M. An almost perfect matching is also maximum, since any subgraph containing all of the vertices must have an odd order, and a matching always has an even order. The following Theorems are covered in any basic Graph Theory text, and since they are not especially helpful in our games, they will be presented without proof:

Theorem 1 *A matching M in G is a maximum matching if and only if there exists no M-augmenting path in G.*

Theorem 2 ***Tutte's Theorem.*** *If $k_o(G) := \{v \in G | v$ is incident with an odd number of edges$\}$, then G has a perfect matching if and only if $k_o(G - S) \leq |S|$, for any proper subset $S \subset V(G)$.*

Proof: See [Gould, pp 193-4]. ∎

Our next two Theorems describes the matching strategy, and why we try to find perfect matchings.

Theorem 3 *If G has a perfect matching, then it in $\mathcal{N}$, no matter where the starting position is.*

Proof: If there is a perfect matching, M then given any initial vertex, Allan moves along the edge in M. Now, Beatrice must start a new edge in M, allowing Allan to stay on that edge. Every time she goes, Beatrice must move on to the beginning point of an edge in M, and thus Allan can always move to the end of that edge. Since Allan has a winning strategy, and Beatrice does not, than G is a first player win, and is thus in $\mathcal{N}$. ∎

Theorem 4 *If G has an almost perfect matching, M, with a hole where the marker is initially placed, then G is a second player win.*

Proof: Allan must move onto the beginning of an edge in M, and Beatrice merely follows the strategy given in Theorem 3, since the game is now in $\mathcal{N}$. Thus Beatrice has matching strategy, and Allan has no good strategy, and so the game is a second player win, and thus $\mathcal{G}(G) = \mathcal{P}$. ∎

Theorem 4 is not as general as Theorem 3, since it depends on where the initial marker is placed. As we will later see, for some move sets there are holes which can never give rise to an almost-perfect matching, while some have an almost perfect matching for any hole, if n is above a certain bound.

Theorem 5 *If G is undirected and can be divided into two independent subgraphs, N and M, which together span G, and M and N are both matchings, then G has a perfect matching.*

Proof: Clearly we can apply the matching to M and N, and since they are independent, each vertex in G is connected to exactly one other vertex, thus making it a perfect matching. ∎

Theorem 5 allows us to explore matching strategies, since if a perfect matching exists, it is a simple strategy which allows Allan to win with little thought. In the first variant of Geography which we will discuss, matching strategies, if they exist are useful, but there are many occasions where matching strategies cannot be used.

4 Kotzig's Nim

The first type of Geography game which we will discuss is *Kotzig's Nim*, or *Modular Nim*. Kotzig's Nim is played on a directed cycle of n vertices. Players can move from vertex u to v if $u = v + a_i$ modulo n, for a_i in move set, A. For the move set of $\{1\}$ or $\{\pm 1\}$, Kotzig's Nim is a trivial game which is determined purely by parity. If n is even, G is in $\mathcal{P}$, and if odd, G is in $\mathcal{N}$. If n and a are relatively prime, the game with move set $\{a\}$ is also determined by parity, and if a and n are not relatively prime, the game has the same outcome as the game $\frac{n}{d}$ with move set $\{\frac{a}{d}\}$, where $d = gcd(a, n)$. These results are simple to find, however the game becomes more complicated when played with a move set of more than one (relatively prime) numbers. If we analyze Kotzig's Nim with an undirected graph, or use move set $\{\pm a, \pm b\}$, which is in fact the same thing, it becomes similar to the version of the line Geography which we will look at in great detail later on.

By taking the Cartesian Product of two cycles of length n and m, we come to the game of Geography played on an $n \times m$ torus. We say that $G = C_n \times C_m$. We then refer to the vertex (i, j), where i is taken modulo n, and j is taken modulo m. We use the move set $\{1\}$, and so there is no need to use an augmented graph. We can think of this graph as an $n \times m$ array of vertices, where edges go left and down, and can wrap around, from the bottom of one column to the top of the same, or around the side. We begin with a theorem of Nowakowsi and Poole. [3]

Theorem 6 *Let $G = C_n \times C_m$. If $n = 2$ or n and m are even then $G \in \mathcal{N}$.*

Proof: If $n = 2$, Allan moves only to change the first coordinate. He does this on the first move, and Beatrice must then move to change the second coordinate. Allan again can change the first coordinate, and since he can always do this, he wins the game, and so it is in $\mathcal{N}$.

If n and m are both even, then Allan always moves from a vertex where the coordinates are of the same parity to one where they are different. Let $M = \{(ab, a(b+1)) | a \equiv b \pmod 2)\}$. M is a perfect matching for G, and thus by Theorem 3, G is in $\mathcal{N}$. ■

Nowakowski and Poole have also shown that for $n = 3$, the value is determined to be $\mathcal{P}$ or $\mathcal{N}$, depending on the value of m modulo 42. Table 1 Gives the values of n modulo 42, for proofs see [3].

n		+6	+12	+18	+30	+36
1	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$
2	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{N}$
3	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$
4	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$
5	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{P}$
6	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{N}$

Table 1: Grundy values for $C_n \times C_3$ taken modulo 42

There are no Matching Strategies in this case, and so other strategies must be developed. Theorem 6 and Table 4 give many result in this variant of Kotzig's Nim, but there are still many cases which have yet to be solved.

5 Line Geography

5.1 The Basics

The game of Line Geography is another variant of the game of Geography which has as its underlying graph an undirected graph with n vertices in a line. We number our vertices $\{0, 1, \ldots, n-1\}$, and we use a move set $\{a, b\}$, where $a < b$. In the augmented graph, two vertices u and v are adjacent if $|v - u| = a$ or b. Each vertex has a degree of two, three, or four. We first try to figure out for the move set $\{a, b\}$, for which even values of n we may apply a matching strategy. To help in this, we use a result from Number Theory:

Theorem 7 ***Sylvester's Theorem*** *If a and b are nonnegative and relatively prime, then there exists Natural s and t such that*

$$as + bt = C, \forall C > ab - a - b \tag{1}$$

Proof: Since a and b are relatively prime, if $a|jb$ then $a|j$. It is then apparent that if $bj \equiv bk (\text{mod } a)$ then $j \equiv k (\text{mod } a)$. Now, $jb + ka$ for

$0 \le j \le a-1$ and any positive k is an equation of the type we are looking for. If we take each of these equations modulo a, we see that for any $n \equiv jb$, if $n \ge jb$, $n = jb + ka$. So we obviously can find a solution to our Diphantine equation in this case. So $(a-1)b$ has a solution, and so do the $a-1$ values before it. And thus the greatest number which cannot be expressed in the required terms is

$$(a-1)b - a = ab - a - b$$

which is as we required. ■

We use this result to figure out some of the games which matching strategies can be used on.

Theorem 8 *If a is in the move set, and $2n = 2a$, then there exists a perfect matching.*

Proof: We create the matching M, where the edges of M are:

$$\{(0,a),(1,a+1),\cdots,(a-1,2a-1)\}$$

It is apparent that each vertex in G is in M, and thus we have a perfect matching. ■

Theorem 9 *For any move set $\{a,b\}$, where $gcd(a,b) = 1$, there exists a C such that there exists a perfect matching in the Graph of size $2n$ for any $n > C$.*

Proof: Theorem 7 ensures us a C such that for all $n > C, n = ja + kb$. Thus $2n = 2ja + 2kb$. Thus G can be broken up into $j+k$ independent subgraphs which together span G, and from Theorem 8, these all have perfect matchings, and from Theorem 5, since they are independent G also has a perfect matching. ■

This Theorem is an excellent beginning to our quest for perfect matchings. Once we determine what C is, we are left with a finite set of possible n values which we need to find a perfect matching. Theorems 8 to 9 give us a bound, but they do not necessarily give us the best bound for C. Before we start looking for specific bounds, however, we will look at some small games, which though trivial, help us to develop tools to find bounds for other move sets.

5.2 Small Games

First we look at the game with move set $\{a\}$, consisting of just one number. When we start on any n, we find that the first move completely determines every other move in the game. If Allan moves from 0 to a, Beatrice cannot move backwards, she must move forwards. Immediately, a few results become apparent.

Theorem 10 *If the move set $A = a$, only graphs of order $2ka$ have a perfect matching.*

Proof: Since we have edges of only one length, we must from the edge $(0, a)$, since 0 has no other vertices besides a adjacent to it. Likewise we must from edges $(1, a+1), \ldots, (a-1, 2a-1)$, which is a grouping of size $2a$. Our result follows. ■

Theorem 11 *With move set $\{1, b\}$, any even graph $2n$ has a perfect matching, and thus the game $2n$ is an N game.*

Proof: Starting at 0, we simply connect each $2i$ with $2i + a$. This is a perfect matching, and since we never used a jump of b, it is a perfect matching regardless of b. ■

One of the things which we keep on insisting upon is relative primality. We take a moment to discuss why this is so important. We start with an example. If we have move set $\{2, 4\}$, we can begin working on a small graph, say $n = 6$. WE can define our graph by the following edges:

$$E(G) = \{(0, 2), (1, 3), (2, 4), (3, 5), (0, 4), (1, 5)\}.$$

We find that our graph is in fact made up of two independent subgraphs, G_1 with vertices $\{1, 3, 5\}$, and three edges connecting these in a triangle, and G_2 with vertices $\{0, 2, 4\}$, and with three edges also in a triangle. One the initial marker is placed, the game is played on only one of these graphs, since there is no path which goes from G_1 to G_2. in fact, the game we are playing on is not $6; \{2, 4\}$, it is $3, \{1, 2\}$.

We can generalize this by observing that if $\gcd(a, b) = d$, and $d|n$, then:

$$n; \{a, b\} = \frac{n}{d}; \{\frac{a}{d}, \frac{b}{d}\} \tag{2}$$

However, if d does not divide n, then:

$$n; \{a, b\} = n'; \{\frac{a}{d}, \frac{b}{d}\} \tag{3}$$

where n' is determined by where the initial marker is placed.

So far, we have only concerned ourselves with perfect matchings which use just edges of length a or length b, in each subgraph. This is strong enough to ensure a C value after which all $2n$ have a perfect matching, but how about values of n which do not break down into segments of $2a$ and $2b$? Our next Theorem will fill in some of the blanks, but first we need some lemmas.

Lemma 1 *For move set $\{a, b\}$, with $a \leq b$, the graph with $n < 2a$ does not have a perfect matching.*

Proof: Our graph does not contain vertex $2a - 1$, since it is the $2a$th vertex, therefore the vertex $a - 1$ is not adjacent to any other vertex, so any matching cannot be perfect. ■

We begin with the case where $a = 2$ and b is odd. Theorem 7 gives us a bound of

$$2(2b - b - 2) = 2b - 4$$

but we can do much better than that.

Lemma 2 *If $b = 1(mod\ 4)$, then for the game $b + 1; \{2, b\}$ there is a perfect matching, and this is the smallest graph with $n \equiv 2(mod\ 4)$ which has a perfect matching.*

Proof: We form the edge $(0, b)$. We than have $4k$ vertices left in the middle, which can be connected as in Theorem 8 with edges of length 2. This is the smallest graph where $n \equiv 2(\text{mod } 4)$ which has a perfect matching, since any smaller cannot have any edges of length b. ■

Lemma 3 *If $b \equiv 3(mod\ 4)$, then then for the game $b + 3; \{2, b\}$ there is a perfect matching, and this is the smallest graph with $n \equiv 2(mod\ 4)$ which has a perfect matching.*

Proof: We form the edges of length b,

$$E_b = \{(0, b), (1, b+1), (2, b+2)\}$$

and again we are left with a multiple of 4 vertices in the middle, which can be connected with edges of length a. Any smaller $n \equiv 2 (\text{mod } 4)$ has no edges of length b, and so cannot have a perfect matching. ■

Theorem 12 *In the game $G = b+1; \{2, b\}$, the best bound is $C = 4\lfloor \frac{b}{4} \rfloor + 2$.*

Proof: Follows from the two Lemmas. If $b \equiv 1(\text{mod } 4)$, then the highest n with no perfect matching is $b + 1 - 4 = b - 3 = 4\lfloor \frac{b}{4} \rfloor + 2$. If $b \equiv 3(\text{mod } 4)$, then the highest n with ne perfect matching is $b+3-4 = b-1 = 4\lfloor \frac{b}{4} \rfloor + 2$. ■

Now we have a complete accounting of the Game $\{2, b\}$, for any value of b odd, and we can continue exploring our game.

5.3 Lowering the Bound

Though we have a C bound, it is not necessarily the lowest bound possible. To this end, we develop some more tools to deal with our move sets, concentrating on the parity of a and b, and on the difference, which we denote j, between them. We will often refer to a move set as $\{a, a+j\}$ instead of $\{a, b\}$, and we refer to out game n in terms of a and b or j, this allows us to find some generalizations which will help us to calculate the lowest bound for specific cases. As we will see, looking at our set $\{a, a+j\}$ modulo j or modulo 2 is a valuable method, but it turns out that the best way to think of our vertices is modulo a.

0	3	6
1	4	7
2	5	

Table 2: Game $8; \{3, 7\}$ shown modulo 3

When starting to look at our Game Graphs, we first begin to think of them as a line with vertices connected with arcs of length a and b. When dealing with large values of n, however, this can get rather tedious, and so we begin stacking our vertices in lines of length a. Thus, two vertices are adjacent if they are horizontally (or vertically depending on how we draw the graph) next to each other, or if they are connected with diagonal line of a slope which depends on j. Table 5.3 shows this method of viewing our graph.

The next Theorem deals with the case of a and b both odd, and helps to cut our C bound in half.

Theorem 13 *For both a and b odd, if $n = a + b$, there is a perfect matching.*

Proof: We look at the matching M with set of all edges:

$$E_a = \{(0, a), (2, a + 2), \dots, (2j, a + 2j), \dots, (b - 1, a + b - 1)\};$$

and

$$E_b = \{(1, b + 1), \dots, (2j + 1, b + 2j + 1), \dots, (a - 2, a + b - 2)\}.$$

E_a is the set of vertices were all of length a and E_b is the set of edges of length b. Now for even $2j$, if $0 \leq 2j \leq b - 1$, then $2j$ is at the beginning of an edge of length a. If $b + 1 \leq 2j \leq a + b - 2$, then $2j$ is at the end of an edge of length b. Thus each even vertex is in exactly one edge. Likewise, for odd $2j - 1$, if $a \leq 2j - 1 \leq a + b - 1$, then $2j - 1$ is at the end of an edge of length a, and if $1 \leq 2j - 1 \leq a - 2$, then $2j - 1$ is at the beginning of an edge of length b. Thus each odd vertex is in exactly one edge, and so G has a perfect matching. ∎

This gives rise to the following Corollary, which follows using simple algebra from our previous theorems:

Corollary 1 *If a and b are odd, the the Graph $2n = aj + bk$ has a perfect matching, if $b < 2n$.*

Proof: If both j and k are even, than the conditions of Theorem 8 are satisfied, since we can set

$$2n = 2j'a + 2k'b$$

where $j' = j2$ and $k' = \frac{k}{2}$ If j and k are both odd, then we have

$$2n = (a + b) + 2j'a + 2k'b$$

where $j' = j - 1$ and $k' = k - 1$. If one of j or k is even, and the other is odd then we have an odd term plus an even term adding up to $2n$, which is clearly ridiculous. ■

Though this is in fact a reasonable bound, there are games which do not satisfy the above equation, such as the game $8; \{3, 7\}$, which has matching with edges $E(M) = \{(0, 7), (1, 4), (2, 5), (3, 6)\}$. (See Table 5.3) In the case where a is odd, the move set $\{a, a + 2\}$ seems to have the property that n has a perfect matching only when $n = aj + bk$, but though it is possible to prove this in any specific case, no proof that it works in each specific case has been found.

5.4 Two Conjectures

Now we can take a different track and start thinking about the case where a and b are not both odd. It is natural to start thinking that if we found such a nice equation for a and b odd, maybe their are similar equations in this case. This question turns out to be much more difficult then in our previous case. In fact, though the two Conjectures below hold for each small case which we will look at in this paper, and for any other cases tried, there is as yet no proof or explanation for the results. The Conjecture 1 is perhaps the more interesting of the two, and so accordingly we begin with it.

Conjecture 1 *For $a, a + j$, j odd, the graph $n = 2a + j - 1$ has exactly one perfect matching.*

Observations 1 1. *This is a symmetric matching.*

2. *In all of these graphs, the vertex $j - 1$ is only incident to one edge, namely $(j - 1, a + j - 1)$, therefore this edge must be in the perfect matching.*

3. *The vertices $\{0, \ldots, a\}$ all have edges which go "forwards".*

We can use these observations to find the actual matching. Furthermore, If we look at a modulo j, we find an interesting result, namely If we look at the set of vertices, $\{0, \ldots, a\}$, we see that the vertices which in M are incident with an edge of length $a+j$ follow a pattern modulo j. This pattern is given in Table 5.4. In the Table, the columns represent the values of j, and the horizontal entries are the value of a modulo j. When $a = 1(\text{mod } j)$, it is in the top row, to make the pattern clearer.

	3	5	7	9	11	13
1(mod j)						1 3 5 7 9 11
					1 3 5 7 9	1 2 3 6 7 10 11
				1 3 5 7	0 3 4 7 8	3 4 5 9 10 11
			1 3 5	2 3 6 7	0 1 5 6 7	0 2 5 7 8 10
		1 3	0 3 4	-	1 4 6 7 9	0 3 6 7 9 10
	1	2 3	3 4 5	4 5 6 7	5 6 7 8 9	6 7 8 9 10 11
	0	0 1	0 1 2	0 1 2 3	0 1 2 3 4	0 1 2 3 4 5
		0 2	1 2 5	-	0 2 3 5 8	1 2 4 5 8
			0 2 4	0 1 4 5	2 3 4 8 9	1 3 4 6 9 11
				0 2 4 6	1 2 5 6 9	1 2 3 6 7 8
					0 2 4 6 8	0 1 4 5 8 9
-1(mod j)						0 2 4 6 8 10

Table 3: Long lines, modulo $a - b$

A few patterns immediately become apparent, but they become clearer if we look at the differences between the entries. Table 4 gives the differences. Both of these Tables are curious, but finding the next column proves to be tricky. Blanks have been left where the two numbers are not relatively prime to each other. Since the next column would be 15, there would be blanks at 3, 5, 6, 9, 10, and 12.

Conjecture 2 *For $a, a + j$, j odd, the graph $n = 2a + j + 1$ has a perfect matching.*

As with Conjecture 1, this conjecture has also not been proven. It suffices to say that for all of the examples which we will look at, the conjecture holds, an so we will use it later on.

5.5 Move Set $\{a, a + 1\}$

So far we have not come up with a bound for the move set $\{a, a + 1\}$ that is better than that which we get from Theorem 7, namely: $2(a(a + 1) - a - a - 1) = 2(a^2 - a - 1)$. In the small examples which we will look at below,

3	5	7	9	11	13
					2 2 2 2 2
				2 2 2 2	1 3 1 3 1
			2 2 2	3 1 3 1	1 1 4 1 1
		2 2	1 3 1	1 4 1 1	2 3 2 1 2
	2	3 1	-	3 2 1 2	3 3 1 2 1
-	1	1 1	1 1 1	1 1 1 1	1 1 1 1 1
-	1	1 1	1 1 1	1 1 1 1	1 1 1 1 1
	2	1 3	-	2 1 2 3	1 2 1 3 3
		2 2	1 3 1	1 1 4 1	2 1 3 3
			2 2 2	1 3 1 3	1 1 4 1 1
				2 2 2 2	1 3 1 3 1
					2 2 2 2 2

Table 4: The differences

this is in fact the best bound, but whether this is true in general is not yet known. However, we can start with small examples, and then use these to eliminate possibilities when we test out specific move sets.

Lemma 4 *For move set $\{a, a+1\}$, $3a$ has no perfect matching.*

$$\begin{array}{ccc}
0 & a & 2a \\
1 & a+1 & 2a+1 \\
\vdots & \vdots & \vdots \\
a-2 & 2a-2 & 3a-2 \\
a-1 & 2a-1 & 3a-1
\end{array}$$

Table 5: A modular array turns out to be the most useful in the case of 3a;{a, a+1}

Proof: See Table 5.5. We are looking at the set of vertices $\{0, 1, \cdots, 3a-2\}$. We start by considering the vertex 0, which in a perfect matching must be incident to the edge $(0, a+1)$, or $(0, a)$.

First we consider the case where $E(M)$ contains the edge $(0, a+1)$. Now we must include the edge $(1, a+2)$, since we cannot form $(1, a+1)$, as $a+1$ is already incident to another edge. This forces $(2, a+3)$, and so inductively we find that our edges are forced to have length of $a+1$, up to $(a-1, 3a)$, leaving the vertices $2a+2, \cdots, 3a-1$ without an edge incident to them.

So we must look at the edge $(0, a)$. Now we must form the edges $(a-1, 2a)$ and $(2a-1, 3a-1)$. Now we must form the edges $(a-2, 2a-2), (a-3, 2a-3), \cdots, (1, a+1)$. Now the vertices $2a+1, \cdots, 3a-1$ have no edges incident with them, and thus there is no perfect matching on this graph. ∎

We apply a similar proof to larger and larger cases, until we can use induction to prove our desired results.

Lemma 5 *The graph* $4a-2; a, a+1$ *does not have a perfect matching.*

0	a	$2a$	$3a$
$\vdots$	$\vdots$	$\vdots$	
$a-3$	$2a-3$	$3a-3$	$4a-3$
$a-2$	$2a-2$	$3a-2$	
$a-1$	$2a-1$	$3a-1$	

Table 6: Vertices can be joined by a horizontal line - or a diagonal of slope 1.

Proof: See Table 5.5. As in Lemma 4, we quickly eliminate the edge (0, a +1), as it cuts our graph into two independent subgraphs, one with $2a+2$ vertices, and one with $2a-4$ vertices. It is apparent from Lemma 1 that $2a-4$ does not have a perfect matching, so we are left with the edge $(0, a)$, and by symmetry we also have the edge $(3a-3, 4a-3)$.

Now, if we had the edge $(a-1, 2a-1)$, it would force $(a-2, 2a-2), \cdots, (1, a+1)$, thus subdividing our graph into graphs of size $2a$ and $2a-2$, the latter of which cannot be solved, from Lemma. Now, We know that we have the edges $(a-1, 2a), (a-3, 2a-2)$, and so we must have $(a-2, 2a-1)$, and know we observe that $3a-1$ is adjacent only to vertices in other matchings, and therefore we do not have a perfect matching. ∎

Lemmas 4 and 5 are fairly simple, but through the proofs we have come up with two important facts about the move set $\{a, a+1\}$, namely, if the edge $(0, a+1)$ is in our matching, we have divided our graph into two subgraphs, one of which is of size $2a + 2$, and the other of order $n - 2a - 2$. We also observed that if we have the edge $(a - 1, 2a - 1)$, we divide our graph into graphs of order $2n$ and $n - 2n$. For our further testing we can immediately eliminate several options.

6 Some Results

	2	3	4	5	6	7	8	9	10
2	-								
3	2	-							
4	-	10	-						
5	2	4	24	-					
6	-	-	-	38	-				
7	6	4	6	18	58	-			
8	-	4	-	18	-	82	-		
9	6	-	12	8	-	40	110	-	
10	-	22	-	-	-	26	-	142	-

Table 7: The lowest bound on the move set $\{a, b\}$

We start out proving that the bounds given in Table 6 are in fact the best bounds for the sets $\{a, b\}$. We have already dealt with the case where $a = 2$, so we can continue with the 3rd row. Note that when a and b are relatively prime to each other we have left a blank.

6.1 Move Set {3, b}

We start with the move set $\{3, 4\}$. Theorem 7 gives us a bound of $2(3^2 - 3 - 1 = 5) = 10$, and so we only have to test 2, 4, 6, 8, and 10. In fact, testing 10 will give us the bound, if 10 does not have a perfect matching, but since this is our first example, and a fairly small one, we will test all of the values. However, Lemma 1 allows us to eliminate 2 and 4, as they are less than 2*3. We know 6 and 8 have perfect matchings from Theorem 8, since they are $2 * 3$ and $2 * 4$ respectively, and thus by Theorem 1 have perfect matchings. $10 = 4 * 3 - 2$, and so Lemma 5 tells us that there is no perfect matching, and thus we know that 10 is the best bound for C.

Next we look at the move set $\{3, 5\}$. Since both 3 and 5 are odd, we can use Theorem 9 to show that a bound is $3 * 5 - 3 - 5 = 7$, and since 7 is odd 6 is our bound, but 6 has a perfect matching, so 4 is our bound, which has no perfect matching, since it is less than 6, by Lemma 1.

We skip 6, since it is not relatively prime to 3, and continue with the move set $\{3, 7\}$. Theorem 9 gives us a bound of $3 \cdot 7 - 3 - 7 = 11$, so we look at 10. We again eliminate 2 and 4 as being too small, and we know 6 has a perfect matching, and from our earlier example we know that 8 also has a matching (see Table 5.3), so all we have to test is 10. The matching with edges:

$$E_3(M) = \{(0, 3), (2, 5), (4, 7), (6, 9)\}$$

and

$$E_7(M) = \{(1, 8)\}$$

is a perfect matching,(see Table 5.3) and so our best bound is 4.

0	3	6	9
1	4	7	
2	3	8	

Table 8: a perfect matching with $\{3, 7\}$

For $\{3, 8\}$ Conjectures 1 and 2 tell us that we can find a perfect matching for 12 and 14, as well as 6 and 16. A double application of Theorem 7 allows us to come up with a bound of 14, and we know that the only other unknown n is 8. 8 has a perfect matching with the edge $(0, 7)$, which leaves us with 6 in the center which have a matching with 3, and since $14 = 8 + 6$, and so by Theorem 5 14 also has a perfect matching, and our bound is 4.

For the final entry, we will look at the move set $\{3, 10\}$. Again, we know that we have a perfect matching for 6 and 20, from Theorem 8 and 12 and 14, from the Conjectures, and so we have a bound of 22, with 8, 10 and 16 as unknowns. Since 22 is comparatively large, we will start with the smaller cases. 8 and 10 can have no edges of length 10, as they are too small. So we begin with 16.

See Table 6.1. If we form the edge $(1, 10)$, we are forced to from the edges $(3, 13)$, $4, 7)$, and $(1, 11)$, and thus 14 is not adjacent to any unmatched vertex, and so this cannot be the proper matching. So we are left with the edges $(0, 3)$ and $(12, 15)$, which force the edges $(6, 19)$, $(2, 5)$, $(8, 11)$, and

0	3	6	9	12	15
1	4	7	10	13	
2	3	8	11	14	

Table 9: There is no perfect matching in the game 16;{3,10}

$(4, 14)$. Now 1 has no vertices adjacent to it, and so there is no perfect matching for this graph. We can use this to simplify the case of $n = 22$.

In the case of 22, If we form the edge $(0, 3)$, then we are forced to form the edges $(7, 17)$, $(10, 20)$, $(0, 3)$, $(13, 16)$, $(6, 9)$, and we observe that 19 is now dead. Then we must from the edges $(0, 10)$ and $(11, 21)$ and the edges $(1, 4)$ and $(17, 20)$, and 14 is now dead. And so there is no perfect matching for 22, and it is our best bound. Now we have completed the third row.

6.2 Move Set {4, b}

We begin discussing the case where $a = 4$ with the move set $\{4, 5\}$. Theorem 9 gives us the bound $2(4 * 5 - 4 - 5) = 22$. If we can show that there is no perfect matching on the graph 22, then we can set this as our bound.

0	4	8	12	16	20
1	5	9	13	17	21
2	6	10	14	18	
3	7	11	15	19	

Table 10: The game 22;{4,5} has no perfect matching

If we from the edge (4, 10), we are forced to form the edges (15, 19), (10, 16), (7, 12), (6, 11), (2, 5) and 1 is dead. If we form (14, 9), the same thing happens, so we must form the edge (14, 19). Now (5, 10) kills 9, and (5, 9) forces (5, 10), (8, 11), and 12 is dead. So (1, 5) and by symmetry (16, 20) and thus we are forced to form (2, 7)) and now 12 is dead, therefore 22 has no perfect matching, and is thus the best bound.

For $\{4, 7\}$, we can use our previous Theorems and Conjectures to see that we can form 8, 10, 12 and 14. Thus 6 is our bound, and as it is less than 8, we know that this is in fact the best bound. Finally we think about the set $\{4, 9\}$. We can form 8, 12, 14 and 18. using our previous theorems, we come up with a bound of 10, which has a perfect matching with $(0, 9)$, leaving 8 in the middle. 8 has a perfect matching, and so $6 < 8$ is our best bound, from Lemma 1, and thus we have finished the fourth row.

6.3 Move Set {5, b}

We begin the fifth row with the move set $\{5, 6\}$. Theorem 9 gives us the bound $C = 2(5 * 6 - 5 - 6) = 38$. In fact, 38 is the best bound.

Next we look at the move set $\{5, 7\}$ Theorem 1 gives us a bound of $C = 5 * 7 - 5 - 7 = 23$. So we look at 22, which is 7 + 3*5, and so has a matching. 20 = 2*10, and so we look at 18.

0	5	10	15
1	6	11	16
2	7	12	17
3	8	13	
4	9	14	

Table 11: No perfect matching for 18;{5, 7}

If we form the edge (0, 7), we are forced to form edges (2, 9), (4, 11) and then 14 is dead. So we must form the edges (0, 5) and (12, 17). If we form (2, 9), we must form (4, 11), and 16 is dead. so we form (2, 7), and (10, 15), thus we form (9, 14) and (4, 11), and 16 is dead. So 18 does not have a perfect matching, and is thus the best bound for $\{5, 7\}$.

For the move set $\{5, 8\}$, Theorem 9 Gives us a bound of 18, and we test that this in fact does not have a perfect matching. If we form the edge (0, 8), we must form (3, 11) and then 16 is dead. So we form (0, 5), and (12, 17), and we are forced to form (4, 9) and (1, 6) and 14 is dead. Thus 18 is indeed the best bound.

For the move set $\{5, 9\}$, Corollary 1 gives us a bound of 31. The largest even n which is not a combination of 5 and 9 is 22. However, this is a combination of 16 and 10, so if we show 16 has a perfect matching, we do not need to worry about 22. In fact, it is apparent that if we form the edges (2, 11) and (4, 13), the rest of the graph has a perfect matching with edges of length 5. Similarly in the case of 12, We form the edge (1, 10) and the rest follows with lines of length 5. So our bound is in fact 8.

We can continue this for the rest of the chart, as the numbers get larger, the proofs become longer and less intuitive. Accordingly, the proof of the rest of the chart is left to the reader to show.

7 Holes and Almost Perfect Matchings

Once we have a handle on games of even degree, we can look at those of odd degree. We know from Theorem 4 that if we can from an almost perfect matching, with the initial vertex as a hole, the Game is in $\mathcal{P}$. However, we must figure out where this hole can be. We start with a, b, both odd.

Theorem 14 *If a and b are both odd, there is no perfect matching with a hole on an odd numbered vertex.*

Proof: Every edge is of odd length, and so it must have as its incident vertices one odd numbered and one even numbered vertex. Thus we are left with a bipartite graph, with even vertices and odd vertices separated. Since our set of vertices goes from 0 to $2j$, some j, We find that there is one more odd numbered vertex, than even, and so by removing another odd numbered one, we are left with two even numbered vertices which cannot be in an edge, and thus we have the desired result. ■

This is helpful, but our results reach further still. For each move set we can start through and find one n value for which there is a hole at each even numbered vertex which gives rise to an almost perfect matching. From this point, we use our previous results in conjunction with Theorem 7 to show that there is a bound above which all of the odd n values have almost perfect matchings for each even numbered vertex.

Further, when we look at the move set $\{a, b\}$ where a and b are not both odd and relatively prime, we can find a bound above which all odd n have possible holes at every vertex. These results are interesting, and through studying both perfect matchings and almost perfect matchings for a given move set, our results come more easily. We intertwine these two tools and find out much more about the sets. However, the study of these holes is much more involved than can be gone into at this time, they shall perhaps be dealt with in a future paper.

8 Conclusion

Line Geography is just one simple variant of the game of Geography, and yet there are still many results about it which have yet to be found, such as the reasoning and proofs behind Conjectures 1 and 2, and most of the results coming from the study of almost perfect matchings. We have, however, developed many tools and ideas which can be used to study further results in this game, and in other versions of Geography. As well, there are many other games on graphs which can be studied in a similar fashion, using a combination of Game and Graph Theory.

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